Sharp L^p estimates for second order Riesz transforms on multiply–connected Lie groups

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Abstract. We study a class of combinations of second order Riesz transforms on Lie groups $\mathbb{G} = \mathbb{G}_x \times \mathbb{G}_y$ that are multiply connected, composed of a discrete abelian component $\mathbb{G}_x$ and a compact connected component $\mathbb{G}_y$. We prove sharp L^p estimates for these operators, therefore generalizing previous results [13][4].

The proof uses stochastic integrals with jump components adapted to functions defined on the semi-discrete set $\mathbb{G}_x \times \mathbb{G}_y$. The analysis shows that Itô integrals for the discrete component must be written in an *augmented* discrete tangent plane of dimension twice larger than expected, and in a suitably chosen discrete coordinate system. Those artifacts are related to the difficulties that arise due to the discrete component, where derivatives of functions are no longer local.

1. Introduction

Sharp L^p inequalities for pairs of differentially subordinate martingales date back to the celebrated work of Burkholder [7] in 1984 where the optimal constant is exhibited. See also from the same author [9][10]. The relation between differentially subordinate martingales and Caldéron-Zygmund operators is known since Gundy–Varopoulos [19]. Starting with a test function f, martingales are built using Brownian motion and Poisson extensions in the upper half space $\mathbb{R}^+ \times \mathbb{R}^n$.

It is shown that the martingale arising from Rf, where R is a Riesz transform in $\mathbb{R}^n$, is a martinagle transform of that arising from f. The two form a pair of martingales with differential subordination and orthognality. Thus, in the case of Riesz tranforms, the optimal L^p constants could be recovered using probabilistic methods. One derives martingale inequalities under hypotheses of strong differential subordination and orthogonality, see Banuelos–Wang [6].

In the case of second order Riesz transforms, the use of heat extensions in the upper half space instead of Poisson extensions originated in Petermichl-Volberg [20] and was used to prove L^p estimates for the second order Riesz transforms based on the results of Burkholder in Nazarov–Volberg [24] as part of their best-at-time estimate for the Beurling-Ahlfors operator, whose real and imaginary parts themselves are second order Riesz transforms. The sharpness of the constant for the real part of the Beurling-Alfhors operator is proved using probabilistic methods in conjunction with a technique by Bourgain in Geiss–Montgomery–Saksman [18]. See also applications in Banuelos–Baudoin [4]

All these martingale inequalities use special functions found in Pichorides [21], Essén [17], when orthogonality is present in addition to differential subordination or Burkholder [7][8][9], when differential subordination is the only hypothesis.

Deterministic proofs of sharp L^p estimates of Caldéron-Zygmund operators that use Burkholder's theorems are available in the literature. The technique of Bellman functions was used in Nazarov–Volberg [24] for an L^p estimate for certain second order Riesz transforms in the euclidean plane as well as in the recent version on discrete abelian groups [13].

The aim of the present work is two–fold. On one hand we want to derive a stochastic proof of the latter result on discrete abelian groups. On the other hand, we want to generalize the estimate to second order Riesz transforms acting on multiply connected Lie groups, built as the cartesian product of a discrete abelian group with a connected compact Lie group. Previous works based on stochastic methods for the analysis of Riesz transforms on connected compact Lie groups are in Arcozzi [1][2], and sharp L^p estimates were proved in this setting in Banuelos–Baudoin [4] for second order Riesz transforms. The novelty of this text is the generalization to the multiply connected setting. In this sense, it is also a generalization of [13], by regarding each point in the discrete abelain group as a Lie group of dimension zero.

1.1. Differential operators and Riesz transforms.

First order derivatives and tangent planes. For a general n–dimensional Riemannian manifold $\mathbb{M}$, the tangent plane $T_y\mathbb{M}$ at a point $y\in\mathbb{M}$ is spanned by a local family of vectors $(Y_1(y),...,Y_n(y))$. Identifying as usual the vector Y_j with the first order derivative $Y_j=\partial/\partial x_j$, we have for a real function f defined on $\mathbb{M}$, the gradient written as a column vector of size n:

$$\boldsymbol{\nabla}_{\mathbb{M}} f(y) := (\partial f/\partial y_1, ..., \partial f/\partial y_n)(y) = (Y_1 f, ..., Y_n f)(y) = \sum_{j=1}^{n} Y_j f(y)\ Y_j(y) \quad \in\ T_y\mathbb{M}.$$

Now for a Lie group $\mathbb{G} := \mathbb{G}_x \times \mathbb{G}_y$ with $\mathbb{G}_x$ a discrete group of order m and $\mathbb{G}_y$ a connected Lie group of dimension n we define the tangent plane $T_z\mathbb{G}$ at a point $z=(x,y)\in(\mathbb{G}_x\times\mathbb{G}_y)=\mathbb{G}$ in three steps.

First, let $\mathbb{G}_y$ be a compact Lie group endowed with a biinvariant Riemannian metric and denote by $\mathfrak{G}_y$ its Lie algebra. Since $\mathbb{G}_y$ is n–dimensional, we can find a family $(Y_j)_{j=1,...,n}$ of n left invariant vector fields that form an orthonormal basis of $\mathfrak{G}_y$. If f is a function defined on $\mathbb{G}_y$, then the partial derivative of f in the direction Y_j at point $y\in\mathbb{G}_y$ is

$$(\partial f/\partial y_j)(y) := (Y_j f)(y) := (\partial_j f)(y)$$

and the gradient accounting for the infinitesimal variations of f about a point $y\in\mathbb{G}_y$ is as before the n–column–vector

$$\boldsymbol{\nabla}_y f(y) := (\partial f/\partial y_1, ..., \partial f/\partial y_n)(y) = (Y_1 f, ..., Y_n f)(y) = \sum_{j=1}^{n} Y_j f(y)\ Y_j(y) \quad \in\ T_y\mathbb{G}_y$$

Second, the discrete component $\mathbb{G}_x$ is of order m and one has m generators $\mathfrak{G}_x=(g_i)_{i=1,...,m}$. At any point $x\in\mathbb{G}_x$, and given a direction $i\in\{1,...,m\}$, one has two choices of discrete derivatives for each generator, namely the right and the left derivatives:

$$(\partial^+ f/\partial x_i)(z) := f(x g_i, y) - f(x, y) := (\partial_i^+ f)(z)$$
$$(\partial^- f/\partial x_i)(z) := f(x g_i^{-1}, y) - f(x, y) := (\partial_i^- f)(z).$$

Comparing with the continuous component, this suggests that the tangent plane $\hat{T}_x\mathbb{G}_x$ at a point x of the discrete group $\mathbb{G}_x$ is actually composed of the "right" tangent plane $T_x^+\mathbb{G}_x$ and the "left" tangent plance $T_x^-\mathbb{G}_x$. We consequently define the *augmented* discrete gradient $\hat{\boldsymbol{\nabla}}_x f(x)$ noted with a *hat*, as the $2m$–vector of $\hat{T}_x\mathbb{G}_x := T_x^+\mathbb{G}_x \times T_x^-\mathbb{G}_x$ accounting for all the local variations of the function f in the direct vicinity of x, that is the $2m$–column–vector

$$\hat{\boldsymbol{\nabla}}_x f(x) := (X_1^+ f, X_2^+ f, ..., X_1^- f, X_2^- f, ...)(x) = \sum_{i=1}^{m}\sum_{\tau=\pm} X_i^\tau f(x)\ X_i^\tau \quad \in\ \hat{T}_x\mathbb{G}_x.$$

We noted the discrete derivatives $X_i^\pm f := \partial_i^\pm f$ and we noted the discrete $2m$–vectors $X_i^\pm$ as the column vectors of $\mathbb{Z}^{2m}$

$$X_i^+ = (0, ..., 1, ..., 0) \times \mathbf{0}_m, \quad X_i^- = \mathbf{0}_m \times (0, ..., 1, ..., 0),$$

where the 1's in $X_i^\pm$ are located at the i–th position of respectively the first or the second m–tuple. Notice that those vectors are independent of the point x.

Finally, for a function f defined on the cartesian product $\mathbb{G} := \mathbb{G}_x \times \mathbb{G}_y$, the (augmented) gradient $\hat{\boldsymbol{\nabla}}_z f(z)$ at the point $z=(x,y)$ is an element of the tangent plane $\hat{T}_z\mathbb{G} := \hat{T}_x\mathbb{G}_x \times T_y\mathbb{G}_y$, that is a $(2m+n)$–column–vector

$$\begin{aligned}\hat{\boldsymbol{\nabla}}_z f(z) &:= \big(\hat{\boldsymbol{\nabla}}_x f(z), \boldsymbol{\nabla}_y f(z)\big) = (X_1^+ f, X_2^+ f, ..., X_1^- f, X_2^- f, ..., Y_1 f, Y_2 f, ...)(z)\\ &= \sum_{i=1}^{m}\sum_{\tau=\pm} X_i^\tau f(z)\ \hat{X}_i^\tau + \sum_{j=1}^{n} Y_j f(z)\ \hat{Y}_j(z)\end{aligned}$$

where $\hat{X}_i^{\tau}$ and $\hat{Y}_j(z)$ are now column vectors of size $(2m+n)$ with obvious definitions.

Riesz transforms. Following [1][2], recall first that for a general Riemannian manifold $\mathbb{M}$ without boundary, one denotes by $\boldsymbol{\nabla}_{\mathbb{M}}$, $\mathrm{div}_{\mathbb{M}}$ and $\Delta_{\mathbb{M}} := \mathrm{div}_{\mathbb{M}}\boldsymbol{\nabla}_{\mathbb{M}}$ the gradient, the divergence and the Laplacian associated with $\mathbb{M}$, respectively. Then $-\Delta_{\mathbb{M}}$ is a positive operator and the vector Riesz transform is defined as the linear operator

$$\boldsymbol{R}_{\mathbb{M}} := \boldsymbol{\nabla}_{\mathbb{M}} \circ (-\Delta_{\mathbb{M}})^{-1/2}$$

acting on $L_0^2(\mathbb{M})$. It follows that if f is a function defined on $\mathbb{M}$ and $y \in \mathbb{M}$ then $\boldsymbol{R}_{\mathbb{M}} f(y)$ is a vector of the tangent plane $T_y\mathbb{M}$.

Similarily on $\mathbb{M} = \mathbb{G}$, we define $\boldsymbol{\nabla}_{\mathbb{M}} := \hat{\boldsymbol{\nabla}}_z$ as before, and then we define the divergence operator as the formal dual, that is $-\mathrm{div}_{\mathbb{M}} = -\widehat{\mathrm{div}}_z := \hat{\boldsymbol{\nabla}}_z^*$, with respect to the following scalar product of $\mathbb{R}^{(2m+n)}$:

$$\big(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\big)_{L^2(\mathbb{R}^{2m+n})} := \frac{1}{2}\sum_{i=1}^{m}\sum_{\pm} a_i^+ b_i^+ + \sum_{j=1}^{n} a_j b_j$$

where

$$\hat{\boldsymbol{a}} := (a_1^+, a_2^+, \ldots, a_1^-, a_2^-, \ldots, a_1, a_2, \ldots) \in \mathbb{R}^{2m+n},$$

and a similar definition holds for $\hat{\boldsymbol{b}}$. The factor one–half is necessary so as to ensure a natural isometry between vectors of $\mathbb{R}^{2m+n}$ and vectors of $\mathbb{R}^{m+n}$. Thanks to the duality relations $(X_i^{\pm})^* = -X_i^{\mp}$ and $Y_j^* = -Y_j$, the divergence operator writes as the $(2m+n)$–row–vector

$$\begin{aligned}
\hat{\boldsymbol{\nabla}}_z^* f(z) &:= \left(\frac{1}{2}\hat{\boldsymbol{\nabla}}_x^* f(z), \boldsymbol{\nabla}_y^* f(z)\right) \\
&= -\frac{1}{2}\sum_{i=1}^{m}\sum_{\tau=\pm} X_i^{-\tau} f(z)\ \hat{X}_i^{\tau,*} - \sum_{j=1}^{n} Y_j f(z)\ \hat{Y}_j^*(z)
\end{aligned}$$

where $\hat{X}_i^{\tau,*}$ and $\hat{Y}_j^*(z)$ are now row vectors of size $(2m+n)$ that are the transpose of $\hat{X}_i^{\tau}$ and $\hat{Y}_j(z)$ respectively. It follows that the Laplacian $-\Delta_{\mathbb{G}}$ is as expected

$$\begin{aligned}
-\Delta_z f(z) &:= \hat{\boldsymbol{\nabla}}_z^* f(z) \cdot \hat{\boldsymbol{\nabla}}_z f(z) = \frac{1}{2}\hat{\boldsymbol{\nabla}}_x^* f(z) \cdot \hat{\boldsymbol{\nabla}}_x f(z) + \boldsymbol{\nabla}_y^* f(z) \cdot \boldsymbol{\nabla}_y f(z) \\
&= -\sum_{i=1}^{m} X_i^- X_i^+ f(z) \quad - \quad \sum_{j=1}^{n} Y_j^2 f(z) \\
&= -\sum_{i=1}^{m} X_i^2 f(z) \quad - \quad \sum_{j=1}^{n} Y_j^2 f(z) \\
&=: (-\Delta_x f)(z) \quad + \quad (-\Delta_y f)(z)
\end{aligned}$$

where we noted $X_i^2 := X_i^+ X_i^- = X_i^- X_i^+$. The Riesz vector $\big(\hat{\boldsymbol{R}}_z f\big)(z)$ is the $(2m+n)$–column–vector of the tangent plane $\hat{T}_z\mathbb{G}$ defined as the linear operator

$$\hat{\boldsymbol{R}}_z f := \big(\hat{\boldsymbol{\nabla}}_z f\big) \circ (-\Delta_z f)^{-1/2}$$

In particular, we have coordinatewise

$$R_i^{\pm} = \partial_i^{\pm} \circ (-\Delta_z)^{-1/2} \quad \text{and} \quad R_j = \partial_j \circ (-\Delta_z)^{-1/2}.$$

1.2. Main results.

In this text, we are concerned with second order Riesz transforms and combinations thereof. We first define the square Riesz transform in the (discrete) direction i to be

$$R_i^2 := R_i^+ R_i^- = R_i^- R_i^+.$$

Then, given $\alpha := ((\alpha_i^x)_{i=1\ldots m}, (\alpha_{jk}^y)_{j,k=1\ldots n}) \in \mathbb{C}^m \times \mathbb{C}^{n\times n}$, we define R_α^2 to be the following combination of second order Riesz transforms:

$$R_\alpha^2 := \sum_{i=1}^m \alpha_i^x R_i^2 + \sum_{j,k=1}^n \alpha_{jk}^y R_j R_k,$$

where the first sum involves squares of discrete Riesz transforms as defined above, and the second sum involves products of continuous Riesz transforms. This combination is written in a condensed manner as the quadratic form

$$R_\alpha^2 = \langle \hat{\boldsymbol{R}}_z, \boldsymbol{A}_\alpha \hat{\boldsymbol{R}}_z \rangle$$

where $\boldsymbol{A}_\alpha$ is the $(2m+n)\times(2m+n)$ block matrix

$$\boldsymbol{A}_\alpha := \begin{pmatrix} \boldsymbol{A}_\alpha^x & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_\alpha^y \end{pmatrix}$$

with

$$\boldsymbol{A}_\alpha^x = \mathbf{diag}(\alpha_1^x, \ldots, \alpha_m^x, \alpha_1^x, \ldots, \alpha_m^x) \in \mathbb{C}^{2m\times 2m}, \quad \boldsymbol{A}_\alpha^y = (\alpha_{jk}^y)_{j,k=1\ldots n} \in \mathbb{C}^{n\times n}.$$

When p and q are conjugate exponents, let $p^* = \max\{p,q\}$. Our main results are

Theorem 1. *Let $\mathbb{G}$ be a Lie group as defined before. Let R_α^2: $L^p(\mathbb{G}, \mathbb{C}) \to L^p(\mathbb{G}, \mathbb{C})$ be a combination of second order Riesz transforms as defined above. This operator enjoys the estimate*

$$\|R_\alpha^2\| \leqslant \|\boldsymbol{A}_\alpha\|_2 \, (p^*-1).$$

The estimate above is sharp when the group $\mathbb{G} = \mathbb{G}_x \times \mathbb{G}_y$ and $\dim(\mathbb{G}_y) + \dim^\infty(\mathbb{G}_x) \geqslant 2$, where $\dim^\infty(\mathbb{G}_x)$ denotes the number of infinite components of $\mathbb{G}_x$.

Notice that $\|\boldsymbol{A}_\alpha\|_2 = \max(\|\boldsymbol{A}_\alpha^x\|_2, \|\boldsymbol{A}_\alpha^y\|_2) = \max(|\alpha_1^x|, \ldots, |\alpha_m^x|, \|\boldsymbol{A}_\alpha^y\|_2)$.

In the case where $\mathbb{G} = \mathbb{G}_x$ only consists of the discrete component, this whas proved in [14][13] using the deterministic Bellman function technique. In the case where $\mathbb{G} = \mathbb{G}_y$ is a connected compact Lie group, this was proved by Banuelos–Baudouin [4] using Brownian motions defined on manifolds and projections of martingale transforms.

In the case where the function f is real valued, we obtain better estimates involving the Choi constants. Compare with Banuelos–Osekowski [5] and with [13].

Theorem 2. *Assume that $a\boldsymbol{I} \leqslant \boldsymbol{A}_\alpha \leqslant b\boldsymbol{I}$ in the sense of quadratic forms, where a, b are real numbers. Then R_α^2: $L^p(\mathbb{G}, \mathbb{R}) \to L^p(\mathbb{G}, \mathbb{R})$ enjoys the norm estimate $\|R_\alpha^2\|_p \leqslant \mathfrak{C}_{a,b,p}$, where these are the Choi constants.*

The Choi constants (see [11]) are not explicit, except $\mathfrak{C}_{-1,1,p} = p^* - 1$. An approximation of $\mathfrak{C}_{0,1,p}$ is known and writes as

$$\mathfrak{C}_{0,1,p} = \frac{p}{2} + \frac{1}{2}\log\left(\frac{1+e^{-2}}{2}\right) + \frac{\beta_2}{p} + \ldots., \quad \text{with} \quad \beta_2 = \log^2\left(\frac{1+e^{-2}}{2}\right) + \frac{1}{2}\log\left(\frac{1+e^{-2}}{2}\right) - 2\left(\frac{e^{-2}}{1+e^{-2}}\right)^2.$$

1.3. Plan of the paper.

In the next subsection, we recall the weak formulations characterizing the combinations of semi-discrete second order Riesz transforms we are interested in. Section 2 is devoted to the stochastic integrals for semi-discrete functions f defined on $\mathbb{G}$, together with their martingale transforms. Finally, the proofs of the main results are given in Section 3.

1.4. Weak formulations.

Let $f\colon \mathbb{G}\to\mathbb{C}$ be given. The heat extension $\tilde{f}(t)$ of f is defined as $\tilde{f}(t):=e^{t\Delta_z}f=:P_tf$. We have therefore $\tilde{f}(0)=f$. The aim of this section is to derive weak formulations for the second order Riesz transforms. We start with the weak formulation of the identity operator $\mathcal{I}$.

Lemma 3. *Assume f and g in $L^2_0(\mathbb{G})$, then*

$$\begin{aligned}
(\mathcal{I}f,g) &:= (f,g)\\
&= 2\int_0^\infty \left\langle \hat{\boldsymbol{\nabla}}_z P_t f, \hat{\boldsymbol{\nabla}}_z P_t g\right\rangle_{L^2(\mathbb{G};T\mathbb{G})}\mathrm{d}t\\
&= 2\int_0^\infty \sum_{z\in\mathbb{G}}\left\{\sum_{i=1}^m\sum_{\tau=\pm}(X_i^\tau P_tf)(z)\ (X_i^\tau P_tg)(z)+\sum_{j=1}^n (Y_jP_tf)(z)\ (Y_jP_tg)(z)\right\}\mathrm{d}t
\end{aligned}$$

and the sums and integrals that arise converge absolutely.

Proof. This classical formula can be obtained by observing that $\mathrm{d}_tP_t=\Delta_zP_t$ and writing the ODE satisfied by $\phi(t):=(P_tf,P_tg)$. □

In order to pass to the weak formulation for the squares of Riesz transforms, we need the following hypothesis and commutation properties.

Hypothesis. We assume everywhere in the sequel:

1. The discrete component $\mathbb{G}_x$ of the Lie group $\mathbb{G}$ is an abelian group
2. The connected component $\mathbb{G}_y$ of the Lie group $\mathbb{G}$ is a compact Lie group

Lemma 4. (Commutation relations) *Assume the Hypothesis above. Then, we have*

$$\begin{aligned}
Y_j\circ\Delta_z &= \Delta_z\circ Y_j\\
X_i^\tau\circ\Delta_z &= \Delta_z\circ X_i^\tau,\quad \tau\in\{+,-,0,2\}
\end{aligned}$$

Proof. Since $\mathbb{G}=\mathbb{G}_x\times\mathbb{G}_y$ is a cartesian product, we have $[Y_j,X_i^\tau]=0$ and as a consequence $[Y_j,\Delta_x]=0$ and $[X_i^\tau,\Delta_y]=0$. Moreover $[Y_j,\Delta_y]=0$ thanks to the existence of a biinvariant metric on the compact Lie group $\mathbb{G}_y$ from which we chose the vector fields Y_j. This yields $[Y_j,\Delta_z]=0$. Finally, since $\mathbb{G}_x$ is abelian we have $[X_i^\tau,\Delta_x]=0$ and therefore $[X_i^\tau,\Delta_z]=0$. □

Lemma 5. *Assume the Hypothesis and the Commutation lemma above. Assume f and g in $L^2_0(\mathbb{G})$, then*

$$\begin{aligned}
(R_\alpha^2 f,g) &= -2\int_0^\infty \left\langle \boldsymbol{A}_\alpha\hat{\boldsymbol{\nabla}}P_tf,\hat{\boldsymbol{\nabla}}P_tg\right\rangle_{L^2(\mathbb{G};\hat{T}\mathbb{G})}\mathrm{d}t\\
&= -2\int_0^\infty\sum_{z\in\mathbb{G}}\left\{\frac{1}{2}\sum_{i=1}^m\sum_{\tau=\pm}\alpha_i^x\,(X_i^\tau P_tf)(z)\ (X_i^\tau P_tg)(z)\right.\\
&\qquad\left. +\quad \sum_{j=1}^n\alpha_{jk}^y\,(Y_jP_tf)(z)\ (Y_kP_tg)(z)\right\}\mathrm{d}t
\end{aligned}$$

and the sums and integrals that arise converge absolutely.

Proof. We apply the previous Lemma to $R_\alpha^2 f$ instead of f and we are left with integrands of the form

$$\begin{aligned}
\left\langle\hat{\boldsymbol{\nabla}}_zP_tR_\alpha^2f,\hat{\boldsymbol{\nabla}}_zP_tg\right\rangle &= \langle(-\Delta_z)P_tR_\alpha^2f,P_tg\rangle\\
&= \sum_{i,\tau}\alpha_i^x\langle(-\Delta_z)P_tR_i^2f,P_tg\rangle+\sum_{j,k}\alpha_{j,k}^y\langle(-\Delta_z)P_tR_jR_kf,P_tg\rangle\\
&= \sum_{i,\tau}\alpha_i^x\left\langle(-\Delta_z)P_tX_i^-(-\Delta_z)^{-1/2}X_i^+(-\Delta_z)^{-1/2}f,P_tg\right\rangle\\
&\qquad+\quad\sum_{j,k}\alpha_{j,k}^y\left\langle(-\Delta_z)P_tX_j(-\Delta_z)^{-1/2}X_k(-\Delta_z)^{-1/2}f,P_tg\right\rangle\\
&= \sum_{i,\tau}\alpha_i^x\ \langle X_i^\pm P_tf,X_i^\pm P_tg\rangle\quad+\quad\sum_{j,k}\alpha_{j,k}^y\ \langle X_jP_tf,X_kP_tg\rangle
\end{aligned}$$

where we used successively the commutation properties of the Laplacian Δ_z with the vector fields and the commutation properties of the vector fields with $P_t = e^{t\Delta_z}$. This yields the desired result. □

2. Stochastic integrals and martingale transforms

In all what follows, we assume that we have a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a càdlàg (i.e. right continuous left limit) filtration $(\mathcal{F}_t)_{t\geqslant 0}$ of sub-σ–algebras of $\mathcal{F}$. We assume as usual that $\mathcal{F}_0$ contains all events of probability zero. All random walks and martingales are adapted to this filtration.

We define below a semi-discrete random walk $\mathcal{Z}_t := (\mathcal{X}_t, \mathcal{Y}_t) \in \mathbb{G}_x \times \mathbb{G}_y$ with generator $L = \Delta_z$. The jump component $\mathcal{X}_t$ is built thanks to compound Poisson jump processes on the discrete set $\mathbb{G}_x$ wheras the continuous component $\mathcal{Y}_t$ involves standard brownian motions on the manifold $\mathbb{G}_y$. Then, Itô's formula ensures that semi-discrete "harmonic" functions $f\colon \mathbb{R}^+ \times \mathbb{G} \to \mathbb{C}$ solving the backward heat equation $(\partial_t + \Delta_z) f = 0$ are actually martingales $M_t^f := f(t, \mathcal{Z}_t)$ for which we define a class of martingale transforms.

Stochastic integrals on Riemannian manifolds and Itô integral. Following Emery [15][16], see also Arcozzi [1][2], we define the Brownian motion $\mathcal{Y}_t$ on $\mathbb{G}_y$, a compact Riemannian manifold, as the process $\mathcal{Y}_t:\ \Omega \to (0,\ T) \times \mathbb{G}_y$ such that for all smooth functions f: $\mathbb{G}_y \to \mathbb{R}$, the quantity

$$f(\mathcal{Y}_t) - f(\mathcal{Y}_0) - \frac{1}{2}\int_0^t (\Delta_y f)(\mathcal{Y}_s)\, \mathrm{d}s =: (I_{\mathrm{d}f})_t \tag{1}$$

is an $\mathbb{R}$–valued continuous martingale. For any adapted continuous process Ψ with values in the cotangent space $T^*\mathbb{G}_y$ of $\mathbb{G}_y$, if $\Psi_t(\omega) \in T^*_{Y_t(\omega)}\mathbb{G}_y$ for all $t \geqslant 0$ and $\omega \in \Omega$, then one can define the *continuous* Itô integral I_Ψ of Ψ as

$$(I_\Psi)_t := \int_0^t \langle \Psi_s, \mathrm{d}\mathcal{Y}_s\rangle$$

so that in particular

$$(I_{\mathrm{d}f})_t = \int_0^t \langle \mathbf{d}_y f(\mathcal{Y}_s), \mathrm{d}\mathcal{Y}_s\rangle$$

The integrand therefore involves the 1–form of $T_y^*\mathbb{G}_y$

$$\mathbf{d}_y f(y) := \sum_j (\partial_j f)(y)\ \mathrm{d}y^j = \sum_j (X_j f)(y)\ X_j^*$$

Discrete random walks and jump processes. We define the *discrete* m–dimensional random walk $\mathcal{X}_t$ on the *discrete* abelian group $\mathbb{G}_x$ as a tuple of the form $\mathcal{X}_t = (\mathcal{X}_t^1, \ldots, \mathcal{X}_t^m)$ where each $\mathcal{X}_t^i$, $1 \leqslant i \leqslant m$ is a compound jump process defined as follows:

i. For any $1 \leqslant i \leqslant m$, let $\mathcal{N}_t^i$ be a càdlàg Poisson process of parameter λ, that is

$$\forall t, \quad \mathbb{P}(\mathcal{N}_t^i = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

The sequence of instants where jumps occur is noted $(T_k^i)_{k\in\mathbb{N}}$, with the convention $T_0^i = 0$.

ii. Let $(\tau_k)_{k\in\mathbb{N}}$ be a sequence of independent Bernouilli variables

$$\forall k, \quad \mathbb{P}(\tau_k = 1) = \mathbb{P}(\tau_k = -1) = 1/2$$

We set

$$\mathcal{N}_t = \sum_{i=1}^m \mathcal{N}_t^i$$

Almost surely the instants of jumps $((T_k^i)_{k\in\mathbb{N}})_{i=1,\ldots,m}$ are disjoints. Let $(T_k)_{k\in\mathbb{N}} = \cup_{i=1}^m (T_k^i)_{k\in\mathbb{N}}$ the sequence of instants of jumps of $\mathcal{N}_t$, and let $i_{N_t}(\omega)$ be the index of the coordinate where the jumps occurs at time t,

$$\mathrm{d}\mathcal{N}_t = \sum_{i=1}^m \mathrm{d}\mathcal{N}_t^i = \mathrm{d}\mathcal{N}_t^{i_{N_t}}$$

The random walk $\mathcal{X}_t$ started at $\mathcal{X}_0 \in \mathbb{G}_x$ is the càdlàg compound Poisson process (see e.g. PROTTER [23], PRIVAULT [22]) defined as

$$\mathcal{X}_t(\omega) := \mathcal{X}_0 + \sum_{k=1}^{N_t} \hat{X}_{i_k(\omega)}^{\tau_k(\omega)}$$

where we used an additive notation for the discrete abelian group. In differential form, we have

$$\forall t, \quad \mathrm{d}\mathcal{X}_t = \hat{X}_{i_{N_t}}^{\tau_{N_t}} \, \mathrm{d}\mathcal{N}_t$$

Stochastic integrals on discrete groups. We recall for the convenience of the reader the derivation of stochastic integrals for jump processes. We will emphasize the fact that the corresponding Itô's formula involves the action of a discrete 1–form written in a well-chosen local coordinate system of the discrete *augmented* cotangent plane (see details below). Let $1 \leqslant k \leqslant \mathcal{N}_t$ and let (T_k, i_k, τ_k) be respectively the instant, the axis and the direction of the k–th jump. We set $T_0 = 0$. Let $f := f(t,x)$, $t \in \mathbb{R}^+$, $x \in \mathbb{G}_x$ a function defined on $\mathbb{R}^+ \times \mathbb{G}_x$. Then

$$\begin{aligned}
f(t,\mathcal{X}_t) &= f(t,\mathcal{X}_t) - f(t,\mathcal{X}_{T_{N_t}}) + \sum_{k=1}^{N_t} \{f(t,\mathcal{X}_{T_k}) - f(t,\mathcal{X}_{T_{k-1}})\} \\
&= f(t,\mathcal{X}_t) - f(t,\mathcal{X}_{T_{N_t}}) + \sum_{k=1}^{N_t} \{f(t,\mathcal{X}_{T_k}) - f(t,\mathcal{X}_{T_k-}) + f(t,\mathcal{X}_{T_k-}) - f(t,\mathcal{X}_{T_{k-1}})\} \\
&= \int_{T_{N_t}}^t (\partial_t f)(s,\mathcal{X}_s)\mathrm{d}s + \sum_{k=1}^{N_t} \left\{ f(t,\mathcal{X}_{T_k}) - f(t,\mathcal{X}_{T_k-}) + \int_{T_{k-1}}^{T_k} (\partial_t f)(s,\mathcal{X}_s)\mathrm{d}s \right\} \\
&= \int_0^t (\partial_t f)(s,\mathcal{X}_s)\mathrm{d}s + \int_0^t (f(s,\mathcal{X}_s) - f(s,\mathcal{X}_{s-}))\,\mathrm{d}\mathcal{N}_s \\
&= \int_0^t (\partial_t f)(s,\mathcal{X}_s)\mathrm{d}s + \sum_{i=1}^m \int_0^t (f(s,\mathcal{X}_s) - f(s,\mathcal{X}_{s-}))\,\mathrm{d}\mathcal{N}_s^i
\end{aligned}$$

At an instant $s = T_k$ of jump, the integrand in the last term writes as

$$\begin{aligned}
(f(s,\mathcal{X}_s) - f(s,\mathcal{X}_{s-}))\,\mathrm{d}\mathcal{N}_s^i &= \left(f\big(s,\mathcal{X}_{s-} + \tau_{N_s}\,\hat{X}_i\big) - f(s,\mathcal{X}_{s-})\right)\mathrm{d}N_s^i \\
&= (X_i^{\tau_{N_s}} f)(\mathcal{X}_{s-})\,\mathrm{d}\mathcal{N}_s^i \\
&= \frac{1}{2}\{(X_i^2 f)(\mathcal{X}_{s-}) \;+\; \tau_{N_s}\,(X_i^0 f)(\mathcal{X}_{s-})\}\,\mathrm{d}\mathcal{N}_s^i
\end{aligned}$$

where we introduced, for all $1 \leqslant i \leqslant m$,

$$\begin{aligned}
X_i^0 &:= (X_i^+ + X_i^-)/2 \\
X_i^2 &:= (X_i^+ - X_i^-)/2.
\end{aligned}$$

Notice that, for any given $1 \leqslant i \leqslant m$, up to a normalisation factor, the system of coordinate (X_i^2, X_i^0) is obtained thanks to a *rotation* of $\pi/4$ of the canonical system of coordinate (X_i^+, X_i^-). Finally,

$$\begin{aligned}
f(t,\mathcal{X}_t) &= \int_0^t (\partial_t f)(s,\mathcal{X}_s)\mathrm{d}s + \frac{1}{2}\sum_{i=1}^m \int_0^t \{(X_i^2 f)(\mathcal{X}_{s-}) \;+\; \tau_{N_s}\,(X_i^0 f)(\mathcal{X}_{s-})\}\,\mathrm{d}\mathcal{N}_s^i \\
&= \int_0^t \left\{(\partial_t f)(s,\mathcal{X}_s) + \frac{\lambda}{2}(\Delta_x f)(s,\mathcal{X}_s)\right\}\mathrm{d}s + \\
&\qquad + \frac{1}{2}\sum_{i=1}^m \int_0^t (X_i^2 f)(s,\mathcal{X}_{s-})\,\mathrm{d}(\mathcal{N}_s^i - \lambda s) + (X_i^0 f)(s,\mathcal{X}_{s-})\,\mathrm{d}\mathcal{X}_s^i \\
&= \int_0^t \left\{(\partial_t f)(s,\mathcal{X}_s) + \frac{\lambda}{2}(\Delta_x f)(s,\mathcal{X}_s)\right\}\mathrm{d}s + \int_0^t \langle \hat{\mathbf{d}} f(s,\mathcal{X}_{s-}), \mathrm{d}\hat{\mathcal{W}}_s\rangle \\
&=: \int_0^t \left\{(\partial_t f)(s,\mathcal{X}_s) + \frac{\lambda}{2}(\Delta_x f)(s,\mathcal{X}_s)\right\}\mathrm{d}s + \left(I_{\hat{\mathbf{d}} f}^x\right)_t,
\end{aligned} \tag{2}$$

where we set $\mathrm{d}\mathcal{X}_s^i := \tau_{N_s} \mathrm{d}\mathcal{N}_s^i$. Here and in the sequel, we take $\lambda = 2$.

Discrete Itô integral. The stochastic integral above shows that Itô formula (1) for continuous processes has a discrete counterpart involving stochastic integrals for jump processes, namely we have the *discrete* Itô integral

$$\left(I_{\hat{\mathbf{d}}f}^x\right)_t \;:=\; \frac{1}{2}\sum_{i=1}^m \int_0^t (X_i^2 f)(s, \mathcal{X}_{s\_})\ \mathrm{d}(\mathcal{N}_s^i - \lambda s) + (X_i^0 f)(s, \mathcal{X}_{s\_})\ \mathrm{d}\mathcal{X}_s$$

This has a more intrinsic expression similar to the continuous Itô integral (1). If we regard the discrete component $\mathbb{G}_x$ as a "discrete Riemannian" manifold, then this discrete Itô integral involves discrete vectors (resp. 1–forms) defined on the *augmented* discrete tangent (resp. cotangent) space $\hat{T}_x\mathbb{G}_x$ (resp. $\hat{T}_x^*\mathbb{G}_x$) of dimension $2m$ defined as

$$\begin{aligned}
\hat{T}_x\mathbb{G}_x &= \mathrm{span}\{X_1^+, X_2^+, ..., X_1^-, X_2^-, ...\}\\
&= \mathrm{span}\{X_1^2, X_2^2, ..., X_1^0, X_2^0, ...\}\\
\hat{T}_x^*\mathbb{G}_x &= \mathrm{span}\{(X_1^+)^*, (X_2^+)^*, ..., (X_1^-)^*, (X_2^-)^*, ...\}\\
&= \mathrm{span}\{(X_1^2)^*, (X_2^2)^*, ..., (X_1^0)^*, (X_2^0)^*, ...\}
\end{aligned}$$

Indeed, let $\mathrm{d}\hat{\mathcal{W}}_s \in \hat{T}_{\mathcal{X}_s}\mathbb{G}_x$ be the vector and $\hat{\mathbf{d}}f \in \hat{T}_{\mathcal{X}_s}^*\mathbb{G}_x$ be the 1–form respectively defined as:

$$\begin{aligned}
\mathrm{d}\hat{\mathcal{W}}_s &= \mathrm{d}(N_s^1 - \lambda s)\ X_1^2 + ... + \mathrm{d}(N_s^m - \lambda s)\ X_m^2 \ +\ \mathrm{d}X_s^1\ X_1^0 + ... + \mathrm{d}X_s^m\ X_m^0\\
\hat{\mathbf{d}}^x f &= X_1^2 f\ (X_1^2)^* + ... + X_m^2 f\ (X_m^2)^* + X_1^0 f\ (X_1^0)^* + ... + X_m^0 f\ (X_m^0)^*
\end{aligned}$$

We have with these notations

$$\left(I_{\hat{\mathbf{d}}f}^x\right)_t \;:=\; \langle \hat{\mathbf{d}}^x f, \mathrm{d}\hat{\mathcal{W}}_s\rangle_{\hat{T}_x^*\mathbb{G}_x \times \hat{T}_x\mathbb{G}_x}$$

where the factor 1/2 is included in the pairing $\langle\cdot,\cdot\rangle_{\hat{T}_x^*\mathbb{G}_x \times \hat{T}_x\mathbb{G}_x}$.

Semi-discrete stochastic integrals. Let finally $\mathcal{Z}_t = (\mathcal{X}_t,\ \mathcal{Y}_t)$ be a semi-discrete random walk on the cartesian product $\mathbb{G} = \mathbb{G}_x \times \mathbb{G}_y$, where $\mathcal{X}_t$ is the random walk above defined on $\mathbb{G}_x$ with generator Δ_x and where $\mathcal{Y}_t$ is the Brownian motion defined on $\mathbb{G}_y$ with generator Δ_y. For $f := f(t, z) = f(t, x, y)$ defined from $\mathbb{R}^+ \times \mathbb{G}$ onto $\mathbb{C}$, we have easily the stochastic integral involving both discrete and continuous parts:

$$f(t, \mathcal{Z}_t) \;=\; \int_0^t \{(\partial_t f)(s, Z_s) + (\Delta_z f)(s, Z_s)\}\, \mathrm{d}s + \left(I_{\hat{\mathbf{d}}^z f}\right)_t$$

where the *semi-discrete* Itô integral writes as

$$\begin{aligned}
\left(I_{\hat{\mathbf{d}}^z f}\right)_t &:= \left(I_{\hat{\mathbf{d}}^x f}\right)_t + (I_{\mathbf{d}^y f})_t\\
&:= \int_0^t \langle \hat{\mathbf{d}}_x f(s, \mathcal{Z}_{s_}), \mathrm{d}\hat{\mathcal{W}}_s\rangle_{\hat{T}_{\mathcal{X}_s}^*\mathbb{G}_x \times \hat{T}_{\mathcal{X}_s}\mathbb{G}_x} + \int_0^t \langle \mathbf{d}_y f(s, \mathcal{Z}_{s_}), \mathrm{d}\mathcal{Y}_s\rangle_{\hat{T}_{\mathcal{Y}_s}^*\mathbb{G}_y \times \hat{T}_{\mathcal{Y}_s}\mathbb{G}_y}
\end{aligned}$$

2.1. Martingale transforms and quadratic covariations.

Martingale transforms. We are interested in martingale transforms allowing us to represent second order Riesz transforms. Let $f(t, z)$ be a solution to the heat equation $\partial_t - \Delta_z = 0$. Fix $T > 0$ and $Z_0 \in \mathbb{G}$. Then define

$$\forall 0 \leqslant t \leqslant T, \quad M_t^{T, Z_0, f} = f(T - t, Z_t).$$

This is a martingale since $f(T - t)$ solves the backward heat equation $\partial_t + \Delta_z = 0$, and we have in terms of stochastic integrals

$$M_t^{f, T, Z_0} = f(T - t, Z_t) = f(T, Z_0) + \int_0^t \langle \hat{\mathbf{d}}_z f(T - s, Z_{s\_}), \mathrm{d}Z_s\rangle$$

Given $\boldsymbol{A}_\alpha$ the $\mathbb{C}^{(2m+n)\times(2m+n)}$ matrix defined earlier, we note M_t^{α,f,T,Z_0} the martingale transform $\boldsymbol{A}_\alpha * M_t^{f,T,Z_0}$ defined as

$$\begin{aligned} M_t^{\alpha,f,T,Z_0} &:= f(T,Z_0) + \int_0^t \left(\boldsymbol{A}_\alpha \hat{\boldsymbol{\nabla}} f(s,Z_{s\_}), \mathrm{d}Z_s\right) \\ &= f(T,Z_0) + \int_0^t \left\langle \hat{\mathbf{d}}_z f(T-s,Z_{s\_}) \boldsymbol{A}_\alpha^*, \mathrm{d}Z_s\right\rangle \end{aligned}$$

where the first integral involves the L^2 scalar product on $\hat{T}_z\mathbb{G} \times \hat{T}_z\mathbb{G}$ and the second integral involves the duality $\hat{T}_z^*\mathbb{G} \times \hat{T}_z\mathbb{G}$. In differential form:

$$\begin{aligned} \mathrm{d}M_t^{\alpha,f,T,Z_0} &= \left(\boldsymbol{A}_\alpha \hat{\boldsymbol{\nabla}} f(s,Z_{s\_}), \mathrm{d}Z_s\right) \\ &= \sum_{i=1}^m \sum_{\pm} \alpha_i^x \left\{(X_i^2 f)(T-t,Z_{t\_})\,\mathrm{d}(N_t^i - \lambda t) + (X_i^0 f)(t,Z_{t\_})\,\mathrm{d}X_t^i\right\} \\ &\qquad + \sum_{j=1}^n \alpha_{j,k}^y\,(X_j f)(T-t,Z_{t\_})\,\mathrm{d}Y_t^k \end{aligned}$$

Quadratic covariation and subordination. We have the quadratic covariations (see PROTTER [23], DELLACHERIE–MEYER [12], or PRIVAULT [22])

$$\begin{aligned} \mathrm{d}[\mathcal{N}^i - \lambda t, \mathcal{N}^i - \lambda t]_t &= \mathrm{d}\mathcal{N}_t^i \\ \mathrm{d}[\mathcal{N}^i - \lambda t, \mathcal{X}^i]_t &= \tau_{N_t}\,\mathrm{d}\mathcal{N}_t^i \\ \mathrm{d}[\mathcal{X}^i, \mathcal{X}^i]_t &= \mathrm{d}\mathcal{N}_t^i \\ \mathrm{d}[\mathcal{Y}^j, \mathcal{Y}^j]_t &= \mathrm{d}t, \end{aligned}$$

the other quadratic covariations being zero. For any two martingales M_t^f and M_t^g defined as above thanks to their respective heat extensions $P_t f$ et $P_t g$, we have the quadratic covariations

$$\begin{aligned} \mathrm{d}[M^f, M^g]_t &= \sum_{i=1}^m (X_i^2 f)(T-t,\mathcal{Z}_{t\_})\,(X_i^2 g)(T-t,\mathcal{Z}_{t\_})\,\mathrm{d}[\mathcal{N}^i - \lambda t, \mathcal{N}^i - \lambda t]_t \\ &\quad + \sum_{i=1}^m (X_i^0 f)(T-t,\mathcal{Z}_{t\_})\,(X_i^0 g)(T-t,\mathcal{Z}_{t\_})\,\mathrm{d}[\mathcal{X}^i, \mathcal{X}^i]_t \\ &\quad + \sum_{i=1}^m (X_i^2 f)(T-t,\mathcal{Z}_{t\_})\,(X_i^0 g)(T-t,\mathcal{Z}_{t\_})\,\mathrm{d}[\mathcal{N}^i - \lambda t, \mathcal{X}^i]_t \\ &\quad + \sum_{i=1}^m (X_i^0 f)(T-t,\mathcal{Z}_{t\_})\,(X_i^2 g)(T-t,\mathcal{Z}_{t\_})\,\mathrm{d}[\mathcal{X}^i, \mathcal{N}^i - \lambda t]_t \\ &\qquad + \sum_{j=1}^n (X_j f)(T-t,\mathcal{Z}_{t\_})\,(X_j g)(T-t,\mathcal{Z}_{t\_})\,\mathrm{d}[\mathcal{Y}^j, \mathcal{Y}^j]_t \\ &= \sum_{i=1}^m \{\ (X_i^+ f)\,(X_i^+ g)(T-t,\mathcal{Z}_{t\_})\ \mathbb{1}(\tau_{N_t} = 1) \\ &\quad + \ (X_i^- f)\,(X_i^- g)(T-t,\mathcal{Z}_{t\_})\ \mathbb{1}(\tau_{N_t} = -1)\ \}\ \mathrm{d}\mathcal{N}_t^i \\ &\qquad + (\boldsymbol{\nabla}_y f, \boldsymbol{\nabla}_y g)(T-t,\mathcal{Z}_{t\_})\ \mathrm{d}t \end{aligned}$$

Differential subordination. Following WANG [25], given two adapted càdlàg Hilbert valued martingales X_t and Y_t, we say that Y_t is differentially subordinate by quadratic variation to X_t if $|Y_0|_{\mathbb{H}} \leqslant |X_0|_{\mathbb{H}}$ and $[Y,Y]_t - [X,X]_t$ is nondecreasing nonnegative for all t. As a consequence, the estimate

$$\begin{aligned} \mathrm{d}[M^{\alpha,f}, M^{\alpha,f}]_t &= \sum_{i=1}^m |\alpha_i^x|^2 \{\ (X_i^+ f)^2(T-t,\mathcal{Z}_{t\_})\ \mathbb{1}(\tau_{N_t} = 1) \\ &\quad + \ (X_i^- f)^2(T-t,\mathcal{Z}_{t\_})\ \mathbb{1}(\tau_{N_t} = -1)\ \}\ \mathrm{d}\mathcal{N}_t^i \\ &\qquad + (\boldsymbol{A}_\alpha^y \boldsymbol{\nabla}_y f, \boldsymbol{A}_\alpha^y \boldsymbol{\nabla}_y f)(T-t,\mathcal{Z}_{t\_})\ \mathrm{d}t \\ &\leqslant \|\hat{\boldsymbol{A}}_\alpha\|_2^2\ \mathrm{d}[M^f, M^f]_t \end{aligned} \tag{3}$$

shows that the martingale transform $Y_t := M_t^\alpha$ is differentially subordinate (by quadratic variation) to the martingale $X_t := \left\| \hat{\boldsymbol{A}}_\alpha \right\|_2 M_t^f$. The following result of Wang [25]:

Theorem 6. (Wang, 1995) *Let X_t and Y_t be two adapted càdlàg Hilbert–valued martingales such that Y_t is differentially subordinate by quadratic covariation to X_t For $1 < p < \infty$,*

$$\|Y_t\|_p \leqslant (p^* - 1) \|X_t\|_p$$

and the constant $p^* - 1$ is best possible. Strict inequality holds when $0 < \|X\|_p < \infty$ and $p \neq 2$,

implies in our situation that

Lemma 7. *Let M_t^f and $M_t^{\alpha, f}$ as defined above. We have*

$$\forall t, \quad \left\| M_t^{\alpha, f} \right\|_p \leqslant \left\| \hat{\boldsymbol{A}}_\alpha \right\|_2 (p^* - 1) \left\| M_t^f \right\|_p.$$

3. Proofs of the main results

Proof. (of Theorem 1) The proof uses the well-known connection between martingale transforms and singular operators, through the use of projection operators. We refer to Gundy–Varopoulos [19] as well as [3][4]. Following the same strategy, the random trajectories $(\mathcal{B}_t)_{-T \leqslant t \leqslant 0}$ defined on the band $[-T, 0] \times \mathbb{G}$ by

$$\mathcal{B}_t := (-t, \mathcal{Z}_t), \quad \mathcal{B}_{-\infty} = (T, \mathcal{Z}_{-T}), \quad -T \leqslant t \leqslant 0, \quad \mathcal{Z}_{-T} \in \mathbb{G}$$

are replaced by random trajectories $(\mathcal{B}_t)_{-\infty \leqslant t \leqslant 0}$ defined on the upper half space $\mathbb{R}^+ \times \mathbb{G}$

$$\mathcal{B}_t := (-t, \mathcal{Z}_t), \quad \mathcal{B}_{-\infty} = (\infty, \mathcal{Z}_{-\infty}), \quad -\infty \leqslant t \leqslant 0, \quad \mathcal{Z}_{-\infty} \in \mathbb{G}.$$

The latter are therefore trajectories starting at time $t = -\infty$ from a point $\mathcal{Z}_{-\infty} \in \mathbb{G}$ chosen at random uniformly, and finishing at time $t = 0$, when hitting the boundary $\mathbb{G}$ of the upper half space. If $f(t) = P_t f$ is as in the previous section, then $M_t^f = f(\mathcal{B}_t)$, $-\infty \leqslant t \leqslant 0$ is a martingale and $M_t^{\alpha, f}$ its martingale transform as defined previously. The Gundy-Varopoulos approach adapted to second order Riesz transforms – see also [4] yields the projection operator $\mathcal{T}^\alpha$ defined as

$$\forall z \in \mathbb{G}, \quad (\mathcal{T}^\alpha f)(z) := \left(M_0^{\alpha, f} \;\middle|\; \mathcal{Z}_0 = z \right),$$

so that using quadratic covariations as above, one observes that

$$\forall g, \quad (\mathcal{T}^\alpha f, g) = -2 \int_0^\infty \left\langle \boldsymbol{A}_\alpha \hat{\boldsymbol{\nabla}} P_t f, \hat{\boldsymbol{\nabla}} P_t g \right\rangle_{L^2(\mathbb{G}; \hat{T}\mathbb{G})} \mathrm{d}t.$$

Thanks to Lemma 5, this means $\mathcal{T}^\alpha = R_\alpha^2$. Thanks to Wang's result [25] (see Theorem 6 and Lemma 7 above) we have easily

$$\|R_\alpha^2 f\|_p = \|\mathcal{T}^\alpha f\|_p \leqslant \|\boldsymbol{A}_\alpha\|_2 (p^* - 1) \|f\|_p,$$

which concludes the proof of Theorem 1. □

The proof of Theorem 2 follows exactly the same procedure. Recall Choi's result [11] for discrete martingales.

Theorem 8. (Choi, 1992) *Let $(\Omega, (\mathcal{F})_{n \in \mathbb{N}}, \mathbb{P})$ a probability space and X_n an adapted real valued martingale. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a predictable sequence taking values in $[0, 1]$. Let $Y := \alpha * X$ be the martingale transform of X defined for almost all $\omega \in \Omega$ as*

$$Y_0(\omega) = X_0(\omega), \quad \text{and} \quad (Y_{n+1} - Y_n)(\omega) = \alpha_n (X_{n+1} - X_n)(\omega).$$

Then there exists a constant $\mathfrak{C}_p$ depending only on p such that $\|Y\|_p \leqslant \mathfrak{C}_p \|X\|_p$ and the estimate is best possible.

The previous result from Choi is only for discrete martingales. For continuous-in-time martingales, we invoke Theorem 1.6 from the paper [5], namely

Lemma 9. (Banuelos–Osekowski, 2012) *Let X_t and Y_t be two real-valued martingales satisfying*

$$\mathrm{d}\left[Y - \frac{a+b}{2}X, Y - \frac{a+b}{2}X\right]_t \leqslant \mathrm{d}\left[\frac{b-a}{2}X, \frac{b-a}{2}X\right]_t$$

for all $t \geqslant 0$. Then for all $1 < p < \infty$, we have $\|Y\|_p \leqslant \mathfrak{C}_p \|X\|_p$.

Proof. (of Theorem 2) The result is now a corollary of Lemma 9 above with $X_t = M_t^f$ and $Y_t = M_t^{\alpha, f}$. It is not difficult to prove that the difference of quadratic variations above writes in terms of a jump part and a continuous part as

$$\begin{aligned}
&\mathrm{d}\left[Y - \frac{a+b}{2}X, Y - \frac{a+b}{2}X\right]_t - \mathrm{d}\left[\frac{b-a}{2}X, \frac{b-a}{2}X\right]_t \\
&= \sum_{i=1}^{m} \sum_{\pm} (\alpha_i^x - a)(\alpha_i^x - b)\ (X_i^{\pm} f)^2(\mathcal{B}_t)\ \mathbb{1}(\tau_{N_t} = \pm 1)\ \mathrm{d}\mathcal{N}_t^i \\
&\quad + \langle (\boldsymbol{A}_\alpha^y - a\boldsymbol{I})(\boldsymbol{A}_\alpha^y - b\boldsymbol{I})\ \boldsymbol{\nabla}_y f(\mathcal{B}_t)\ ,\ \boldsymbol{\nabla}_y f(\mathcal{B}_t)\rangle\ \mathrm{d}t,
\end{aligned}$$

which is nonpositive since we assumed $a\boldsymbol{I} \leqslant \boldsymbol{A}_\alpha \leqslant b\boldsymbol{I}$. This proves the result. □

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